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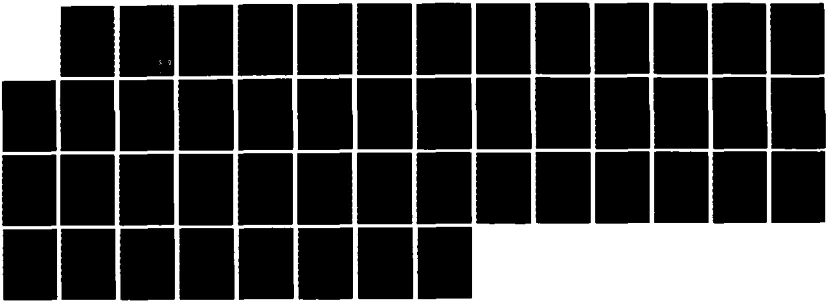

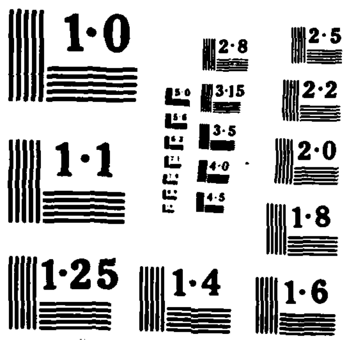
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THE MULTIVARIATE HAZARD CONSTRUCTION

by

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Abstract

A new representation, called the total hazard construction, of dependent random variables by means of independent exponential random variables is introduced. Conditions which imply association of nonnegative random variables are found using this construction. Furthermore, new conditions which imply stochastic ordering between two nonnegative random vectors are obtained. These strengthen previous results of the authors. Further applications in reliability theory and in simulation are indicated.

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1. Introduction.

Let T be an absolutely continuous random variable to be thought of as a lifetime of a device. If F and f are, respectively, the distribution and the density functions of T then $\bar{F} \equiv 1 - F$, $\Lambda \equiv -\log \bar{F}$ and $\lambda = f/\bar{F}$ are, respectively, the survival, hazard and hazard rate functions of T. It is easy to verify that if X is standard (that is, mean 1) exponential random variable then

$$(1.1) \qquad \hat{T} \equiv \Lambda^{-1}(X)$$

satisfies

$$(1.2) \qquad \hat{T} \stackrel{st}{=} T$$

where '$\stackrel{st}{=}$' denotes equality in law.

Equations (1.1) and (1.2) suggest that study of Λ can shed light and understanding on the stochastic behaviour of T. One purpose of this paper is to introduce and study a multivariate analog of (1.1) and (1.2). This is done in Sections 2 and 3 where it is shown how to transform a set of independent standard exponential random variables $X_1, \ldots, X_n$ into a random vector $\underline{\hat{T}}$ which is stochastically equal to a given random vector $\underline{T}$. The transformation uses multivariate hazard functions (to be defined in Sections 2 and 3) and will be called the total hazard construction.

In Section 4 we list conditions under which the transformation $(X_1, \ldots, X_n) \to \underline{\hat{T}}$ is increasing (in this paper 'increasing' stands for 'nondecreasing' and 'decreasing' for 'nonincreasing') in each of the X_i's. Under these conditions, then, the random variables $\hat{T}_1, \ldots, \hat{T}_n$ (and hence

$T_1,\ldots,T_n$) are associated in the sense of Esary, Proschan and Walkup (1967).

Let S and T be two absolutely continuous random variables with hazard functions $Q(w) \equiv -\log P\{S>w\}$ and $R(w) \equiv -\log P\{T>w\}$. Then, using the same standard exponential random variable X, one can apply (1.1) and (1.2) to obtain $\hat{S}$ and $\hat{T}$, defined on the same probability space, such that

$$(1.3) \qquad \hat{S} \equiv Q^{-1}(X) \overset{st}{=} S,\ \hat{T} \equiv R^{-1}(X) \overset{st}{=} T.$$

If

$$(1.4) \qquad Q(w) \geq R(w),\ w \geq 0,$$

then, from (1.3), $S \overset{st}{=} \hat{S} \leq \hat{T} \overset{st}{=} T$. Thus we see that (1.4) implies stochastic ordering of S and T. In Section 5 we obtain a multivariate extension of this result, again using the total hazard construction of Sections 2 and 3. Further applications are given in Section 6.

A random variable S is said to be <u>stochastically smaller</u> than a random variable T (denoted $S \overset{st}{\leq} T$) if $P\{S > u\} \leq P\{T > u\}$ for every u. A random vector $\underline{S} = (S_1,\ldots,S_n)$ is said to be stochastically smaller than a random vector $\underline{T} = (T_1,\ldots,T_n)$ [denoted $\underline{S} \overset{st}{\leq} \underline{T}$] if

$$(1.5) \qquad g(\underline{S}) \overset{st}{\leq} g(\underline{T})$$

for every increasing Borel measurable real function g. A function g is called increasing if $(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n)$ implies $g(x_1,\ldots,x_n) \leq g(y_1,\ldots,y_n)$ where $(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n)$ means $x_i \leq y_i$, $i = 1,\ldots,n$. It is well known that $\underline{S} \overset{st}{\leq} \underline{T}$ if and only if

$$Eg(\underline{S}) \leq Eg(\underline{T})$$

for every increasing Borel measurable real function g for which the expectations exist. Also $\underline{S} \overset{st}{\leq} \underline{T}$ if and only if

$$P\{\underline{S} \in U\} \leq p\{\underline{T} \in U\}$$

for every Borel set U which has an increasing indicator function.

2. The total hazard construction: bivariate case.

Consider two nonnegative random variables T_1 and T_2 with absolutely continuous joint distribution function F, joint density function f and joint survival function $\overline{F}$ defined by $\overline{F}(t_1,t_2) \equiv P\{T_1 > t_1, T_2 > t_2\}$. The conditional hazard rate of T_i at time t, given that $T_{3-i} > t$ is defined as

$$\text{(2.1)} \qquad \lambda_i(t) \equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_i \leq t + \Delta t | T_1 > t, T_2 > t\}$$

$$= [\overline{F}(t,t)]^{-1}\left(- \frac{\partial}{\partial t_i} \overline{F}(t_1,t_2)\Big|_{t_1=t_2=t}\right), \ t \geq 0, \ i = 1,2.$$

Given that $T_2 = t_2$, the conditional hazard rate of T_1 at time $t > t_2$ is defined as

$$\text{(2.2)} \qquad \lambda_1(t|t_2) \equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_1 \leq t + \Delta t | T_1 > t, T_2 = t_2\}$$

$$= - \frac{f(t,t_2)}{\frac{\partial}{\partial t_2} \overline{F}(t,t_2)}, \ t \geq t_2 .$$

(Substituting t and t_2 in the right hand side of (2.2) may yield 0/0. Here and in the remainder of the paper such a ratio is interpreted as 0.) Similarly define

$$(2.3)\qquad \lambda_2(t|t_1) \equiv -\frac{f(t_1,t)}{\frac{\partial}{\partial t_1}\bar{F}(t_1,t)}, \quad t > t_1 .$$

The total hazard accumulated by time t by the random variable T_i, given that $\min(T_1,T_2) > t$, is defined by

$$(2.4)\qquad \Lambda_i(t) \equiv \int_0^t \lambda_i(u)du, \quad t > 0, \quad i = 1,2.$$

Given that $T_2 = t_2$ and that $T_1 > t_2$, the total hazard accumulated by the random variable T_1 during the time interval $[t_2,t_2+t)$ is defined by

$$(2.5)\qquad \Lambda_1(t|t_2) \equiv \int_{t_2}^{t_2+t} \lambda_1(u|t_2)du, \quad t_2 > 0, \quad t > 0.$$

Similarly

$$(2.6)\qquad \Lambda_2(t|t_1) = \int_{t_1}^{t_1+t} \lambda_2(u|t_1)du, \quad t_1 > 0, \quad t > 0.$$

Note that with this definition,

$$(2.7)\qquad \frac{\partial}{\partial t}\Lambda_1(t|t_2) = \lambda_1(t_2+t|t_2), \quad t_2 > 0, \quad t > 0,$$

$$(2.8)\qquad \frac{\partial}{\partial t}\Lambda_2(t|t_1) = \lambda_2(t_1+t|t_1), \quad t_1 > 0, \quad t > 0.$$

The <u>total hazard accumulated by T_1 by the time it failed</u> is defined as

$\Lambda_1(T_1)$ if $T_1 < T_2$ and as $\Lambda_1(T_2) + \Lambda_1(T_1 - T_2 | T_2)$ if $T_1 > T_2$. Similarly define the <u>total hazard accumulated by T_2 by the time it failed.</u>

Define the inverse functions

$$\Lambda_i^{-1}(x) \equiv \inf\{t > 0: \Lambda_i(t) > x\}, \; i = 1,2, \; x > 0,$$
$$\Lambda_1^{-1}(x|t_2) \equiv \inf\{t > 0: \Lambda_1(t|t_2) > x\}, \; x > 0, \; t_2 > 0,$$
$$\Lambda_2^{-1}(x|t_1) \equiv \inf\{t > 0: \Lambda_2(t|t_1) > x\}, \; x > 0, \; t_1 > 0,$$

and consider the functions $a_1: R_+^2 \to R$ and $a_2: R_+^2 \to R$ define as follows: On $\{(x_1,x_2) \in R_+^2: \Lambda_1^{-1}(x_1) < \Lambda_2^{-1}(x_2)\}$,

$$a_1(x_1,x_2) \equiv \Lambda_1^{-1}(x_1), \tag{2.9}$$
$$a_2(x_1,x_2) \equiv \Lambda_1^{-1}(x_1) + \Lambda_2^{-1}(x_2 - \Lambda_2(\Lambda_1^{-1}(x_1)) | \Lambda_1^{-1}(x_1)), \tag{2.10}$$

and on $\{(x_1,x_2) \in R_+^2: \Lambda_1^{-1}(x_1) > \Lambda_2^{-1}(x_2)\}$

$$a_1(x_1,x_2) \equiv \Lambda_2^{-1}(x_2) + \Lambda_1^{-1}(x_1 - \Lambda_1(\Lambda_2^{-1}(x_2)) | \Lambda_2^{-1}(x_2)), \tag{2.11}$$
$$a_2(x_1,x_2) \equiv \Lambda_2^{-1}(x_2) \; . \tag{2.12}$$

Motivated by the fact (see Remark 2.2 below) that the total hazards accumulated by T_1 and T_2 by the time they failed, are independent standard exponential random variables we introduce the following <u>total hazard construction:</u>

Let X_1 and X_2 be independent standard exponential random variables and consider the following transformation:

$$\begin{pmatrix} \hat{T}_1 \\ \hat{T}_2 \end{pmatrix} \equiv \begin{pmatrix} a_1(X_1,X_2) \\ a_2(X_1,X_2) \end{pmatrix} . \tag{2.13}$$

Theorem 2.1. Let $(\hat{T}_1,\hat{T}_2)$ be defined as in (2.13) where X_1 and X_2 are independent standard exponential random variables. Then

$$(2.14) \qquad (\hat{T}_1,\hat{T}_2) \overset{st}{\geq} (T_1,T_2) .$$

Proof. Let $\hat{\lambda}_i$, $\hat{\lambda}_1(\cdot|t_2)$, and so on, be the analogs of λ_i, $\lambda_1(\cdot|t_2)$, and so on, defined in (2.1) - (2.3) with $\hat{T}_i$ replacing T_i, $i = 1,2$. We will show that

$$(2.15) \qquad \hat{\lambda}_1(t) = \lambda_1(t), \; t \geq 0,$$

$$(2.16) \qquad \hat{\lambda}_2(t) = \lambda_2(t), \; t \geq 0,$$

$$(2.17) \qquad \hat{\lambda}_1(t|t_2) = \lambda_1(t|t_2), \; t \geq t_2 \geq 0,$$

$$(2.18) \qquad \hat{\lambda}_2(t|t_1) = \lambda_2(t|t_1), \; t \geq t_1 \geq 0,$$

and the result then follows from the fact that

$\lambda_1(\cdot)$, $\lambda_2(\cdot)$, $\lambda_1(\cdot|t_2)$, $\lambda_2(\cdot|t_1)$, $t_1 \geq 0$, $t_2 \geq 0$ uniquely determine f (see, for example, the explicit formulas in Cox (1972) or in Shaked and Shanthikumar (1984a)) and that

$\hat{\lambda}_1(\cdot)$, $\hat{\lambda}_2(\cdot)$, $\hat{\lambda}_1(\cdot|t_2)$, $\hat{\lambda}_2|\cdot(t_1)$, $t_1 \geq 0$, $t_2 \geq 0$, determine the joint density of $\hat{T}_1$ and $\hat{T}_2$ in a similar manner.

To show (2.15) notice that

$\{\hat{T}_1 > t, \hat{T}_2 > t\} = \{\Lambda_1^{-1}(X_1) > t, \Lambda_2^{-1}(X_2) > t\} = \{X_1 > \Lambda_1(t), X_2 > \Lambda_2(t)\}$.

Hence, for $t \geq 0$,

$$\begin{aligned}
&P\{t < \hat{T}_1 \leq t + \Delta t|\hat{T}_1 > t, \hat{T}_2 > t\} \\
&= P\{t < \Lambda_1^{-1}(X_1) \leq t + \Delta t|X_1 > \Lambda_1(t), X_2 > \Lambda_2(t)\} \\
&= P\{\Lambda_1(t) < X_1 \leq \Lambda_1(t + \Delta t)|X_1 > \Lambda_1(t), X_2 > \Lambda_2(t)\}
\end{aligned}$$

$$= P\{X_1 < \Lambda_1(t + \Delta t) - \Lambda_1(t)\}$$
$$= 1 - \exp\{-(\Lambda_1(t + \Delta t) - \Lambda_1(t))\}.$$

Thus

$$\hat{\lambda}_1(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < \hat{T}_1 < t + \Delta t | \hat{T}_1 > t, \hat{T}_2 > t\}$$

$$= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left(1 - \exp\{-(\Lambda_1(t+\Delta t) - \Lambda_1(t))\}\right)$$

$$= \frac{d}{dt} \Lambda_1(t) = \lambda_1(t)$$

and (2.15) follows. The proof of (2.16) is the same.

To prove (2.17) note that for $t > t_2 > 0$,

$$\{\hat{T}_1 > t, \hat{T}_2 = t_2\}$$
$$= \{\Lambda_2^{-1}(X_2) + \Lambda_1^{-1}(X_1 - \Lambda_1(\Lambda_2^{-1}(X_2)) | \Lambda_2^{-1}(X_2)) > t, \Lambda_2^{-1}(X_2) = t_2\}$$
$$= \{t_2 + \Lambda_1^{-1}(X_1 - \Lambda_1(t_2) | t_2) > t, \Lambda_2^{-1}(X_2) = t_2\}$$
$$= \{X_1 > \Lambda_1(t_2) + \Lambda_1(t - t_2 | t_2), \Lambda_2^{-1}(X_2) = t_2\} .$$

Thus, for $t > t_2 > 0$,

$$P\{t < \hat{T}_1 < t + \Delta t | \hat{T}_1 > t, \hat{T}_2 = t_2\}$$
$$= P\{t < \Lambda_2^{-1}(X_2) + \Lambda_1^{-1}(X_1 - \Lambda_1(\Lambda_2^{-1}(X_2)) | \Lambda_2^{-1}(X_2)) < t + \Delta t$$
$$| X_1 > \Lambda_1(t_2) + \Lambda_1(t - t_2 | t_2), \Lambda_2^{-1}(X_2) = t_2\}$$
$$= P\{t - t_2 < \Lambda_1^{-1}(X_1 - \Lambda_1(t_2) | t_2) < t + \Delta t - t_2$$
$$| X_1 > \Lambda_1(t_2) + \Lambda_1(t - t_2 | t_2), \Lambda_2^{-1}(X_2) = t_2\}$$
$$= P\{\Lambda_1(t_2) + \Lambda_1(t - t_2 | t_2) < X_1 < \Lambda_1(t_2) + \Lambda_1(t - t_2 + \Delta t | t_2)$$
$$| X_1 > \Lambda_1(t_2) + \Lambda_1(t - t_2 | t_2), \Lambda_2^{-1}(X_2) = t_2\}$$
$$= P\{X_1 < \Lambda_1(t - t_2 + \Delta t | t_2) - \Lambda_1(t - t_2 | t_2)\} .$$

Hence, for $t > t_2 > 0$,

$$\hat{\lambda}_1(t|t_2) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}(1-\exp\{-(\Lambda_1(t-t_2+\Delta t|t_2) - \Lambda_1(t-t_2|t_2))\})$$

$$= \frac{d}{dt}\Lambda_1(t-t_2|t_2) \underset{(2.7)}{=} \lambda_1(t|t_2)$$

and (2.17) follows. The proof of (2.18) is the same. ∥

Remark 2.2. From (2.13) one easily obtains: On $\{\hat{T}_1 < \hat{T}_2\}$, $X_1 = \Lambda_1(\hat{T}_1)$, $X_2 = \Lambda_2(\hat{T}_1) + \Lambda_2(\hat{T}_2-\hat{T}_1|\hat{T}_1)$, and on $\{\hat{T}_1 > \hat{T}_2\}$, $X_1 = \Lambda_1(\hat{T}_2) + \Lambda_1(\hat{T}_1-\hat{T}_2|\hat{T}_2)$, $X_2 = \Lambda_2(\hat{T}_2)$. Thus X_1 and X_2 are the total hazards accumulated by $\hat{T}_1$ and $\hat{T}_2$ by the times they failed. Since $(T_1, T_2) \overset{st}{=} (\hat{T}_1, \hat{T}_2)$, it follows that the total hazards accumulated by T_1 and T_2 by the time they failed are independent standard exponential random variables (see also Remark 3.2).

Examp'^ 1 (Freund (1961) distribution). The Freund distribution provides a model in which the exponenti:l residual life o^f one component depends on the working status of another component. It has the density

$$(2.19)\qquad f(t_1,t_2) = \alpha\beta'\exp\{-(\alpha+\beta-\beta')t_1 - \beta' t_2\} \quad \text{if} \quad 0 < t_1 < t_2 ,$$
$$= \alpha'\beta\exp\{-\alpha' t_1 - (\alpha+\beta-\alpha')t_2\} \quad \text{if} \quad 0 < t_2 < t_1 ,$$

where α, α', β, β' are nonnegative parameters. For this distribution

$$\Lambda_1(t) = \alpha t, \; t > 0,$$
$$\Lambda_2(t) = \beta t, \; t > 0 ,$$

$$\Lambda_1(t|t_2) = \alpha' t, \; t \geq 0, \; t_2 \geq 0 ,$$
$$\Lambda_2(t|t_1) = \beta' t, \; t \geq 0, \; t_1 \geq 0 .$$

If (T_1,T_2) has the joint distribution (2.19) then from (2.13) and (2.14) it follows that it has the same distribution as $(\hat{T}_1,\hat{T}_2)$ where

$$(2.20) \qquad \begin{aligned} \hat{T}_1 &= \alpha^{-1}X_1 && \text{if} \quad \alpha^{-1}X_1 \leq \beta^{-1}X_2 , \\ &= \beta^{-1}X_2 + (\alpha')^{-1}(X_1-\beta^{-1}\alpha X_2) && \text{if} \quad \alpha^{-1}X_1 > \beta^{-1}X_2 , \\ \hat{T}_2 &= \alpha^{-1}X_1 + (\beta')^{-1}(X_2-\alpha^{-1}\beta X_1) && \text{if} \quad \alpha^{-1}X_1 \leq \beta^{-1}X_2 , \\ &= \beta^{-1}X_2 && \text{if} \quad \alpha^{-1}X_1 > \beta^{-1}X_2 , \end{aligned}$$

and X_1 and X_2 are independent standard exponential random variables. Representation (2.20) can be rewritten as

$$(2.21) \qquad \begin{aligned} \hat{T}_1 &= \alpha^{-1}X_1 && \text{if} \quad \alpha^{-1}X_1 \leq \beta^{-1}X_2 , \\ &= (\alpha')^{-1}X_1 + \beta^{-1}(1 - \tfrac{\alpha}{\alpha'})X_2 && \text{if} \quad \alpha^{-1}X_1 > \beta^{-1}X_2 , \\ \hat{T}_2 &= (\beta')^{-1}X_2 + \alpha^{-1}(1 - \tfrac{\beta}{\beta'})X_1 && \text{if} \quad \alpha^{-1}X_1 \leq \beta^{-1}X_2 , \\ &= \beta^{-1}X_2 && \text{if} \quad \alpha^{-1}X_1 > \beta^{-1}X_2 . \end{aligned}$$

Representation (2.21) is identical to (7) of Shaked (1984).

The example will be continued later.

<u>Example 2</u> (bivariate Pareto). Let (T_1,T_2) have the joint survival function

$$\bar{F}(t_1,t_2) = (1+t_1+t_2)^{-1} , \; t_1 \geq 0, \; t_2 \geq 0 .$$

It is not hard to verify that in this case

$$(2.22) \qquad \Lambda_1(t) = \Lambda_2(t) = \frac{1}{2}\log(1+2t), \quad t > 0\ ,$$

$$(2.23) \qquad \Lambda_1(t|t_2) = 2\log\left(1+\frac{t}{1+2t_2}\right), \quad t > 0,\ t_2 > 0\ ,$$

$$(2.24) \qquad \Lambda_2(t|t_1) = 2\log\left(1+\frac{t}{1+2t_1}\right), \quad t > 0,\ t_1 > 0\ .$$

Some algebra shows (using (2.13) and (2.14)) that (T_1,T_2) has the same distribution as $(\hat{T}_1,\hat{T}_2)$ where

$$\hat{T}_1 = \frac{1}{2}(e^{2X_1}-1) \qquad \text{if} \quad X_1 < X_2\ ,$$

$$= \exp\{\tfrac{1}{2}X_1 + \tfrac{3}{2}X_2\} - \tfrac{1}{2}(e^{2X_2}+1) \qquad \text{if} \quad X_1 > X_2\ ,$$

$$\hat{T}_2 = \exp\{\tfrac{3}{2}X_1 + \tfrac{1}{2}X_2\} - \tfrac{1}{2}(e^{2X_1}+1) \qquad \text{if} \quad X_1 < X_2\ ,$$

$$= \frac{1}{2}(e^{2X_2}-1) \qquad \text{if} \quad X_1 > X_2\ ;$$

here X_1 and X_2 are independent standard exponential random variables.

This example will be continued later.

3. The total hazard construction: multivariate case.

Consider a random vector $\underline{T} = (T_1,\ldots,T_n)$, $n \geq 2$, with absolutely continuous joint distribution function. In this section we describe the total hazard construction of a random vector $\underline{\hat{T}} = (\hat{T}_1,\ldots,\hat{T}_n)$ such that $\underline{T} \stackrel{st}{=} \underline{\hat{T}}$. The construction will be described in n steps numbered 1 through n. In Step 1 an index j_1 is chosen at random from $\{1,2,\ldots,n\}$ and then $\hat{T}_{j_1}$ is determined. Upon entering Step k, $2 \leq k \leq n$, the random variables $\hat{T}_{j_1},\ldots,\hat{T}_{j_{k-1}}$ have already been determined where $J \equiv \{j_1,\ldots,j_{k-1}\} \subset \{1,\ldots,n\}$. In Step k an index j_k is chosen at random from $\bar{J} \equiv \{1,\dots,n\} - J$ and then $\hat{T}_{j_k}$ is determined.

We need to extend and slightly modify the notation of Section 2.

For $J = \{j_1,\dots,j_k\} \subset \{1,\dots,n\}$ let $\underline{t}_J$ denote $(t_{j_1},\dots,t_{j_k})$. If $\bar{J} = \{i_1,\dots,i_{n-k}\}$ then $\underline{t}_{\bar{J}}$ denotes $(t_{i_1},\dots,t_{i_{n-k}})$. Let $\underline{e} = (1,\dots,1)$. The length of $\underline{e}$ will vary from one formula to another, but it will be always possible to determine it from the expression in which $\underline{e}$ appears.

For $J \subset \{1,\dots,n\}$ and $i \in \bar{J}$ let $\lambda_i(t|\underline{T}_J = \underline{t}_J, \underline{T}_{\bar{J}} > t\underline{e})$ denote the conditional hazard rate of T_i at time t given that $\underline{T}_J = \underline{t}_J$ and that $\underline{T}_{\bar{J}} > t\underline{e}$ where $t > \bigvee_{j\in J} t_j \equiv \max\{t_j : j \in J\}$. If $J = \emptyset$ then $\bigvee_{j\in J} t_j \equiv 0$. Formally, for $i \in \bar{J}$,

$$(3.1)\qquad \lambda_i(t|\underline{T}_J = \underline{t}_J, \underline{T}_{\bar{J}} > t\underline{e})$$
$$\equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_i < t + \Delta t|\underline{T}_J = \underline{t}_J, \underline{T}_{\bar{J}} > t\underline{e}\},\ t > \bigvee_{j\in J} t_j$$

(J may be empty). The absolute continuity of $\underline{T}$ ensures that this limit exists. To save space we sometimes supress the condition $\underline{T}_{\bar{J}} > t\underline{e}$ and just write $\lambda_i(t|\underline{T}_J = \underline{t}_J, \cdot)$ but the reader should keep in mind that '$\cdot$' means $\underline{T}_{\bar{J}} > t\underline{e}$ with t being the same as the first argument of λ_i . The function $\lambda_i(\cdot|\underline{T}_J = \underline{t}_J, \cdot)$ will be of interest for us only on the (random) interval $(\max_{j\in J} T_j, \min_{i\in\bar{J}} T_i]$, however, to avoid a discussion of such random hazard rate functions [such a discussion can be found in Arjas (1981)] we do not emphasize this point here. Note however that $\lambda_i(t|\underline{T}_J = \underline{t}_J, \cdot)$ is well defined for every $t > \bigvee_{j\in J} t_j$.

For $i \in \bar{J}$ the total hazard accumulated by T_i during the time interval $[\bigvee_{j\in J} t_j, \bigvee_{j\in J} t_j + t)$, $t > 0$, is defined by

$$(3.2)\qquad \Lambda_i(t|\underline{T}_J = \underline{t}_J) \equiv \int_{\bigvee_{j\in J} t_j}^{\bigvee_{j\in J} t_j + t} \lambda_i(u|\underline{T}_J = \underline{t}_J, \cdot)du,\ t > 0,\ i \in \bar{J} .$$

When $J = \emptyset$, $\Lambda_i(t|\underline{T}_J = \underline{t}_J)$ will be simply denoted by $\Lambda_i(t)$.

We will introduce now a notation for the total hazard accumulated by T_i by time t. Fix $t > 0$ and suppose that it is given that $T_{j_1},\ldots,T_{j_{k-1}}$ $(k > 1)$ failed at times $t_{j_1},\ldots,t_{j_{k-1}}$, respectively $(t_{j_1}<\ldots<t_{j_{k-1}} < t)$ and that all the other T_j's are alive at time t. For $i \notin \{j_1,\ldots,j_{k-1}\}$, denote

$$\begin{aligned}\psi_{i|j_1,\ldots,j_{k-1}}(t|t_{j_1},\ldots,t_{j_{k-1}}) &\equiv \Lambda_i(t_{j_1}) \\ &+ \sum_{\ell=2}^{k-1} \Lambda_i(t_{j_\ell} - t_{j_{\ell-1}}|T_{j_1} = t_{j_1},\ldots,T_{j_{\ell-1}} = t_{j_{\ell-1}}) \\ &+ \Lambda_i(t-t_{j_{k-1}}|T_{j_1} = t_{j_1},\ldots,T_{j_{k-1}} = t_{j_{k-1}}) .\end{aligned}$$

Also denote (corresponding to the case $k = 1$)

$$\psi_i(t) \equiv \Lambda_i(t), \ t > 0.$$

Note that for $t_{j_1} < t_{j_2}<\ldots<t_{j_{k-1}}$ and $i \notin \{j_1,\ldots,j_{k-1}\}$,

$$\begin{aligned}\psi_{i|j_1,\ldots,j_{k-1}}(t_{j_{k-1}}|t_{j_1},\ldots,t_{j_{k-1}}) &= \Lambda_i(t_{j_1}) \\ &+ \sum_{\ell=2}^{k-1} \Lambda_i(t_{j_\ell} - t_{j_{\ell-1}}|T_{j_1} = t_{j_1},\ldots,T_{j_{\ell-1}} = t_{j_{\ell-1}}) .\end{aligned}$$

The <u>total hazard accumulated by T_i by the time it failed</u>, given that T_i was the k-th T_j to fail and that $T_{j_1},\ldots,T_{j_{k-1}}$ failed before T_i, is $\psi_{i|j_1,\ldots,j_{k-1}}(T_i|T_{j_1},\ldots,T_{j_{k-1}})$.

Define the inverse functions

$$\Lambda_i^{-1}(x) = \inf\{t \geq 0: \Lambda_i(t) \geq x\}, \; i = 1,2,\ldots,n, \quad x \geq 0,$$

and for nonempty $J \subset \{1,\ldots,n\}$, $\underline{t}_J \geq 0\underline{e}$ and $i \in \bar{J}$,

$$\Lambda_i^{-1}(x|\underline{T}_J = \underline{t}_J) = \inf\{t \geq 0: \Lambda_i(t|\underline{T}_J = \underline{t}_J) \geq x\}, \; x \geq 0.$$

Motivated by the fact (see Remark 3.2 below) that the total hazards accumulated by $T_1,\ldots,T_n$ by the time they failed, are independent standard exponential random variables, we introduce and study the following <u>total hazard construction:</u>

Let $X_1,\ldots,X_n$ be independent standard exponential random variables.

<u>Step 1.</u> Let j_1 be the (random) index (which, by absolute continuity, is unique with probability 1) such that $\Lambda_{j_1}^{-1}(X_{j_1}) = \min\{\Lambda_i^{-1}(X_i): i = 1,\ldots,n\}$ and define

$$(3.3) \qquad \hat{T}_{j_1} = \Lambda_{j_1}^{-1}(X_{j_1}) \;.$$

<u>Step k</u> $(k = 2,\ldots,n)$. Given that Steps $1,2,\ldots,k-1$ resulted in $\hat{T}_{j_1} = t_{j_1},\ldots,\hat{T}_{j_{k-1}} = t_{j_{k-1}}$ let $J = \{j_1,\ldots,j_{k-1}\}$. Let j_k be the (random) index (which, by absolute continuity, is unique with probability 1) such that

$$\Lambda_{j_k}^{-1}[X_{j_k} - \psi_{j_k|j_1,\ldots,j_{k-1}}(t_{j_{k-1}}|t_{j_1},\ldots,t_{j_{k-1}})|T_{j_1} = t_{j_1},\ldots,T_{j_{k-1}} = t_{j_{k-1}}]$$

$$= \min_{i\in\bar{J}}\{\Lambda_i^{-1}[X_i - \psi_{i|j_1,\ldots,j_{k-1}}(t_{j_{k-1}}|t_{j_1},\ldots,t_{j_{k-1}})|T_{j_1} = t_{j_1},\ldots,T_{j_{k-1}} = t_{j_{k-1}}]\}.$$

It is easy to verify, by induction, that the arguments of Λ_i^{-1} and $\Lambda_{j_k}^{-1}$, in the above expression, are nonnegative. Having chosen the (random) index j_k as described above, define

$$(3.4.\text{i})\quad \hat{T}_{j_k} = \hat{T}_{j_{k-1}} + \Lambda_{j_k}^{-1}[X_{j_k} - \psi_{j_k|j_1,\ldots,j_{k-1}}(\hat{T}_{j_{k-1}}|\hat{T}_{j_1},\ldots,\hat{T}_{j_{k-1}})|T_{j_1} = t_{j_1},\ldots,T_{j_{k-1}} = t_{j_{k-1}}].$$

More explicitly,

$$(3.4.\text{ii})\quad \hat{T}_{j_k} = \hat{T}_{j_{k-1}} + \Lambda_{j_k}^{-1}[X_{j_k} - \Lambda_{j_k}(\hat{T}_{j_1}) - \sum_{\ell=2}^{k-1} \Lambda_{j_k}(\hat{T}_{j_\ell} - \hat{T}_{j_{\ell-1}}|T_{j_1} = \hat{T}_{j_1},\ldots,T_{j_{\ell-1}} = \hat{T}_{j_{\ell-1}})|\ T_{j_1} = \hat{T}_{j_1},\ldots,T_{j_{k-1}} = \hat{T}_{j_{k-1}}].$$

For example, if $n = 3$ and $\Lambda_1^{-1}(X_1) < \Lambda_2^{-1}(X_2)$, $\Lambda_1^{-1}(X_1) < \Lambda_3^{-1}(X_3)$ and $\Lambda_2^{-1}[X_2 - \Lambda_2(\Lambda_1^{-1}(X_1))|T_1 = \Lambda_1^{-1}(X_1)] < \Lambda_3^{-1}[X_3 - \Lambda_3(\Lambda_1^{-1}(X_1))|T_1 = \Lambda_1^{-1}(X_1)]$ then

$$(3.5)\qquad \hat{T}_1 = \Lambda_1^{-1}(X_1),$$

$$(3.6)\qquad \hat{T}_2 = \hat{T}_1 + \Lambda_2^{-1}[X_2 - \Lambda_2(\hat{T}_1)|T_1 = \hat{T}_1] = \hat{T}_1 + \Lambda_2^{-1}[X_2 - \psi_{2|1}(\hat{T}_1|\hat{T}_1)|T_1 = \hat{T}_1],$$

$$(3.7)\qquad \hat{T}_3 = \hat{T}_2 + \Lambda_3^{-1}[X_3 - \Lambda_3(\hat{T}_1) - \Lambda_3(\hat{T}_2 - \hat{T}_1|T_1 = \hat{T}_1)|T_1 = \hat{T}_1,\ T_2 = \hat{T}_2] = \hat{T}_2 + \Lambda_3^{-1}[X_3 - \psi_{3|1,2}(\hat{T}_2|\hat{T}_1,\hat{T}_2)|T_1 = \hat{T}_1,\ T_2 = \hat{T}_2].$$

<u>Theorem 3.1.</u> Let $\underline{\hat{T}}$ be as defined in (3.3) and (3.4) where $X_1,\ldots,X_n$ are independent standard exponential random variables. Then

$$(3.8)\qquad \underline{\hat{T}} \overset{st}{\geq} \underline{T}.$$

To prove Theorem 3.1 define $\hat{\lambda}_i$ analogously to the definition of λ_i in (3.1) with $\hat{\underline{T}}$ replacing $\underline{T}$. Then all that is needed to complete the proof (see, e.g., Shaked and Shanthikumar (1984a)) is to show that for all $J \subset \{1,\ldots,n\}$, $i \in J$, $t > \bigvee_{j \in J} t_j$, we have

$$(3.9) \qquad \hat{\lambda}_i(t|\hat{\underline{T}}_J = \underline{t}_J, \cdot) = \lambda_i(t|\underline{T}_J = \underline{t}_J, \cdot) .$$

The proof of (3.9) is similar to the proofs of (2.15) - (2.18) but is notationally more involved. We omit the details.

Remark 3.2. It can be shown, using (3.3) and (3.4) that, for every permutation $(j_1,\ldots,j_n)$ of $(1,\ldots,n)$, on $\{\hat{T}_{j_1} < \hat{T}_{j_2} < \ldots < \hat{T}_{j_n}\}$ we have

$$\begin{aligned} X_{j_1} &= \psi_{j_1}(\hat{T}_{j_1}) , \\ X_{j_k} &= \psi_{j_k|j_1,\ldots,j_{k-1}}(\hat{T}_{j_k}|\hat{T}_{j_1},\ldots,\hat{T}_{j_{k-1}}), \quad k = 2,\ldots,n . \end{aligned}$$

thus $X_1,\ldots,X_n$ are the total hazards accumulated by $\hat{T}_1,\ldots,\hat{T}_n$ by the times they failed. Since $\underline{T} \overset{st}{=} \hat{\underline{T}}$ it follows that the total hazards accumulated by $T_1,\ldots,T_n$ by the times they failed are independent standard exponential random variables. This fact generalizes Theorem 2.2 of Schechner (1984). It also follows from Section 4.5 of Aalen and Hoem (1978) and Proposition 2.2.11 of Jacobsen (1982).

Remark 3.3. It should be emphasized that the total hazard construction (3.3) and (3.4) is theoretically and practically different than the following well known standard construction (see, e.g., Law and Kelton (1982), p. 268 or Rubinstein (1981), p. 59):

Let $U_1, U_2, \ldots, U_n$ be independent uniform [0,1] random variables and let $\underline{T} = (T_1, \ldots, T_n)$ be an absolutely continuous random vector. Define

$$(3.10)\quad T_1' = \inf\{t_1 : P\{T_1 > t_1\} > U_1\},$$

$$(3.11)\quad T_k' = \inf\{t_k : P\{T_k > t_k | T_1 = T_1', \ldots, T_{k-1} = T_{k-1}'\} > U_k\},\ k = 2,3,\ldots,n.$$

Then

$$(3.12)\quad (T_1', \ldots, T_n') \stackrel{st}{=} (T_1, \ldots, T_n).$$

Although the construction defined by (3.3) and (3.4) is different than the one defined by (3.10) and (3.11), the results which follow from (3.3) and (3.4) have analogs which follow from (3.10) and (3.11). These analogs will be noted throughout the sequel.

4. An application: association of random variables.

4.1. The bivariate case.

Let T_1 and T_2 be nonnegative absolutely continuous random variables as in Section 2 and let $\hat{T}_1$ and $\hat{T}_2$ be defined as in (2.13). Since $(T_1, T_2) \stackrel{st}{=} (\hat{T}_1, \hat{T}_2)$ we will not distinguish in this section between $\underline{T}$ and $\underline{\hat{T}}$ and just write $\underline{T}$.

If the functions a_1 and a_2 defined in (2.9) - (2.12) are increasing in each argument when the other argument is held fixed then from (2.13) it follows that T_1 and T_2 are associated in the sense of Esary, Proschan and Walkup (1967). Association is a property which yields important probability inequalities and is particularly useful in reliability theory (see, e.g., Barlow and Proschan (1975)). Thus it is of interest to find conditions which imply that a_1 and a_2 are increasing.

Theorem 4.1. If for all $y > 0$, $u > 0$, $b > 0$, $i = 1,2$,

$$(4.1) \qquad \Lambda_i(y+u) - \Lambda_i(y) + \Lambda_i(b|y+u) < \Lambda_i(u+b|y)$$

then T_1 and T_2 are associated.

Remark 4.2. Intuitively, for $i = 1$ say, Condition (4.1) says that the larger T_2 is (compare $T_2 = y$ to $T_2 = y + u$) the smaller is the potential hazard that can be accumulated by T_1 by the time $y + u + b$ (see Figure 4.1). Thus, roughly speaking, the larger T_2 is the larger T_1 is and so the association of T_1 and T_2 is not surprising. For a similar result see Arjas and Norros (1984).

Proof of Theorem 4.1. We just have to show that $a_1(x_1,x_2)$ and $a_2(x_1,x_2)$ increase in x_1 and x_2. The result then follows from Esary, Proschan and Walkup (1967).

Consider $a_1(x_1,x_2)$. It is easy to see that, for a fixed x_2, $a_1(x_1,x_2)$ increases in x_1. Thus one just has to show that, for a fixed x_1, the function $a_1(x_1,x_2) = \Lambda_2^{-1}(x_2) + \Lambda_1^{-1}[x_1 - \Lambda_1(\Lambda_2^{-1}(x_2))|\Lambda_2^{-1}(x_2)]$ increases in $x_2 \in [0, \Lambda_2(\Lambda_1^{-1}(x_1))]$. Denote $t = \Lambda_2^{-1}(x_2)$, $t_1 = \Lambda_1^{-1}(x_1)$. We need to show then that, for each $t_1 > 0$,

$$(4.2) \qquad \tilde{a}(t_1,t) \equiv t + \Lambda_1^{-1}[\Lambda_1(t_1) - \Lambda_1(t)|t]$$

increases in $t \in [0, t_1]$.

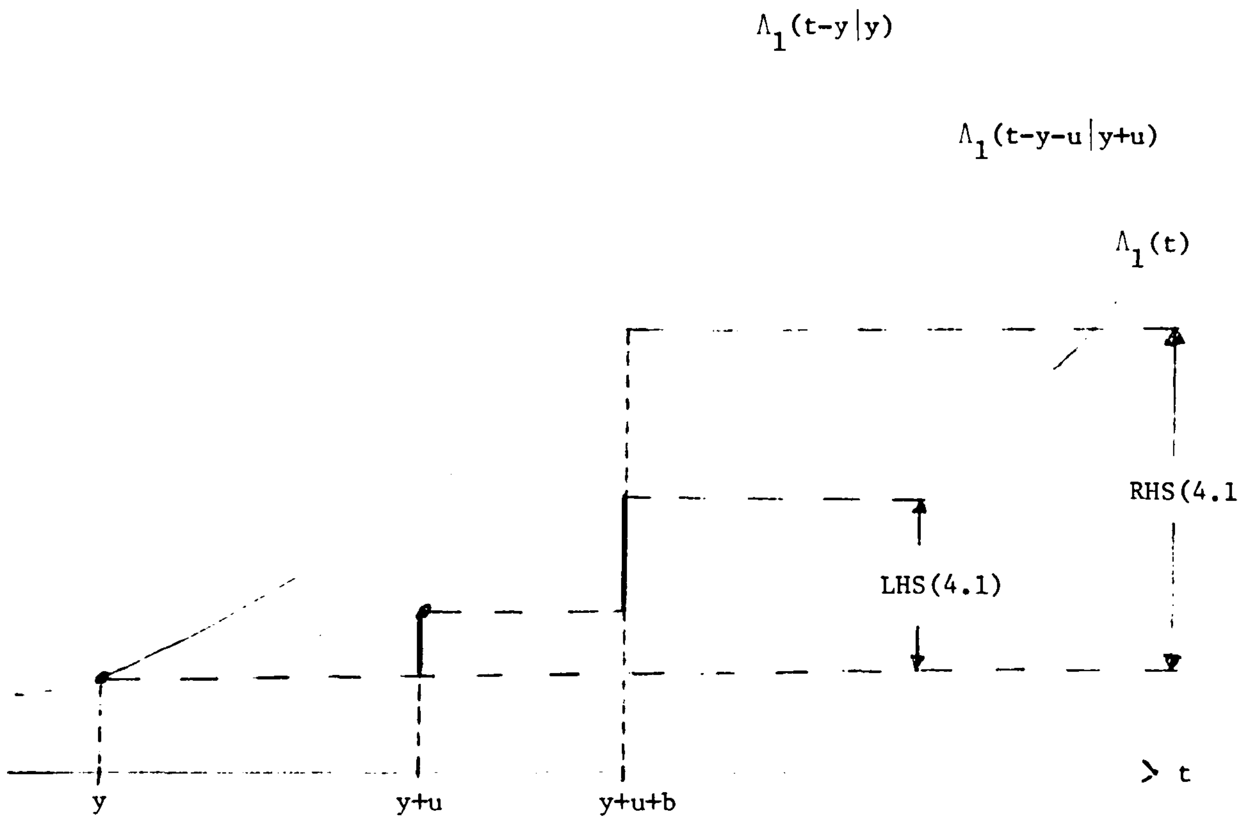

Figure 4.1. Illustration of (4.1).

Rewrite (4.2) as $\Lambda_1(t_1) - \Lambda_1(t) = \Lambda_1[\tilde{a}(t_1,t) - t|t]$. Similarly, if $u > 0$ and $t + u < t_1$, then
$\Lambda_1(t_1) - \Lambda_1(t+u) = \Lambda_1[\tilde{a}(t_1,t+u) - t - u|t + u]$. Thus

$$(4.3) \qquad \Lambda_1(t+u) - \Lambda_1(t) + \Lambda_1[\tilde{a}(t_1,t+u) - t - u|t + u] = \Lambda_1[\tilde{a}(t_1,t) - t|t].$$

But, by (4.1) [here LHS = left hand side, RHS = right hand side],
LHS (4.3) $\leq \Lambda_1[\tilde{a}(t_1,t+u) - t|t]$. Thus,
$\Lambda_1[\tilde{a}(t_1,t) - t|t] \leq \Lambda_1[\tilde{a}(t_1,t+u) - t|t]$. But, for each t, $\Lambda_1(b|t)$ is increasing in b. Hence $\tilde{a}(t_1,t) \leq \tilde{a}(t_1,t+u)$, that is, $\tilde{a}(t_1,t)$ increases in $t \in [0,t_1]$.

Similarly it can be shown that $a_2(x_1,x_2)$ increases in $x_1 \in [0, \Lambda_1(\Lambda_2^{-1}(x_2))]$.∎

Condition (4.1) can be written by means of derivatives (if they are well defined):

<u>Theorem 4.3.</u> If for $i = 1,2$,

$$(4.4) \qquad \lambda_i(y) + \frac{\partial}{\partial a}\Lambda_i(b|a)\Big|_{a=y} \leq \lambda_i(y+b|y), \; b \geq 0, \; y \geq 0$$

(provided the derivatives in (4.4) are well defined) then T_1 and T_2 are associated.

<u>Proof</u>. Rewrite (4.1) as: for $i = 1, 2$, $y \geq 0$, $b \geq 0$,

$$(4.5) \quad [\Lambda_i(y+u) - \Lambda_i(y)] + [\Lambda_i(b|y+u) - \Lambda_i(b|y)] \leq \Lambda_i(b+u|y) - \Lambda_i(b|y), \; u \geq 0.$$

Dividing (4.5) by $u > 0$ and letting $u \downarrow 0$ one obtains (4.4) from (4.5). To obtain (4.5) from (4.4) integrate (4.4) with respect to the dummy variable y. ‖

Example 1 (continued). From (2.21) it is easily seen that if

(4.6) $\alpha < \alpha'$ and $\beta < \beta'$

then $\hat{T}_1$ and $\hat{T}_2$ are increasing functions of X_1 and X_2. Hence if (T_1,T_2) has the Freund distribution with parameters satisfying (4.6) then T_1 and T_2 are associated. This result has been obtained also in Shaked (1984).

Example 2 (continued). Differentiating (2.22) - (2.24) one obtains

$$\lambda_i(y) = \frac{1}{1+2y}, \quad i = 1,2,$$
$$\lambda_i(y+b|y) = \frac{2}{1+2y+b}, \quad i = 1,2,$$
$$\frac{\partial}{\partial a} \Lambda_i(b|a)\Big|_{a=y} = \frac{-4b}{(1+2y)(1+2y+b)}, \quad i = 1,2.$$

It is not hard now to verify (4.4). Hence if (T_1,T_2) has the bivariate Pareto distribution then T_1 and T_2 are associated.

This result is not surprising. Shaked (1977) has shown that the multivariate logistic distribution of Malik and Abraham (1973) has some positive dependence properties. Since the multivariate Pareto distribution is a simple transformation of the multivariate logistic distribution it follows that also the multivariate Pareto distribution has some positive dependence properties. It is not hard to find other representations of T_1 and T_2 as incresing functions of independent random variables.

<u>Example 3</u> (bivariate Gumbel exponential distribution). Let T_1 and T_2 have the bivariate survival function

$$\bar{F}(t_1,t_2) = \exp\{-t_1-t_2-\theta t_1 t_2\},\ t_1 > 0,\ t_2 > 0,$$

where $\theta \in [0,1]$ is a fixed parameter. Here

$$\Lambda_1(t) = \Lambda_2(t) = t + \frac{1}{2}\theta t^2,\ t > 0 \tag{4.7}$$

$$\Lambda_1(u|t) = \Lambda_2(u|t) = u + \theta tu - \log\frac{1+\theta t+\theta u}{1+\theta t},\ t > 0,\ u > 0. \tag{4.8}$$

The inverses Λ_1^{-1} and Λ_2^{-1} do not have as simple expressions as in Examples 1 and 2 but it is still not hard to check (4.4). Differentiating (4.7) and (4.8) one obtains

$$\lambda_i(y) = 1 + \theta y \quad,\ i = 1,2,$$

$$\lambda_i(y+b|y) = 1 + \theta y - \frac{\theta}{1+\theta b+\theta y},\ i = 1,2,$$

$$\frac{\partial}{\partial a}\Lambda_i(b|a)\Big|_{a=y} = \theta b - \frac{\theta}{1+\theta b+\theta y} + \frac{\theta}{1+\theta y},\ i = 1,2.$$

Substituting these in (4.4) it is seen that (4.4) does not hold. Thus we cannot show that T_1 and T_2 are associated. In fact T_1 and T_2 are not associated. This follows from the fact (Johnson and Kotz (1972), p. 262) that they are negatively correlated.

4.2. <u>The multivariate case.</u>

Let $\underline{T} = (T_1,\ldots,T_n)$, $n \geq 3$, be a nonnegative absolutely continuous random vector as in Section 3 and let $\hat{\underline{T}}$ be defined as in (3.3) and (3.4). As in Section 4.1 we will not distinguish between $\underline{T}$ and $\hat{\underline{T}}$ and just write $\underline{T}$.

To clarify the general condition in Theorem 4.4 below, consider the case $n = 3$. Following (3.5) - (3.7), for $\underline{x} = (x_1,x_2,x_3)$ which satisfy $\Lambda_1^{-1}(x_1) < \min(\Lambda_2^{-1}(x_2),\Lambda_3^{-1}(x_3))$ and $\Lambda_2^{-1}(x_2-\Lambda_2(\Lambda_1^{-1}(x_1))|T_1 = \Lambda_1^{-1}(x_1)) < \Lambda_3^{-1}(x_3-\Lambda_3(\Lambda_1^{-1}(x_1))|T_1 = \Lambda_1^{-1}(x_1))$ let

$$\begin{aligned}
a_1(x_1,x_2,x_3) &\equiv \Lambda_1^{-1}(x_1),\\
a_2(x_1,x_2,x_3) &\equiv a_1(\underline{x}) + \Lambda_2^{-1}[x_2 - \psi_{2|1}(a_1(\underline{x})|a_1(\underline{x}))|T_1 = a_1(\underline{x})]\\
&= a_1(\underline{x}) + \Lambda_2^{-1}[x_2 - \Lambda_2(a_1(\underline{x}))|T_1 = a_1(\underline{x})],\\
a_3(x_1,x_2,x_3) &\equiv a_2(\underline{x}) + \Lambda_3^{-1}[x_3-\psi_{3|1,2}(a_2(\underline{x})|a_1(\underline{x}),a_2(\underline{x}))|T_1 = a_1(\underline{x}),T_2 = a_2(\underline{x})]\\
&= a_2(\underline{x}) + \Lambda_3^{-1}[x_3 - \Lambda_3(a_1(\underline{x}))\\
&\quad - \Lambda_3(a_2(\underline{x}) - a_1(\underline{x})|T_1 = a_1(\underline{x}))|T_1 = a_1(\underline{x}),\ T_2 = a_2(\underline{x})].
\end{aligned}$$

Clearly, on the given domain, each a_i increases in x_1, each of a_2 and a_3 increases in x_2 and a_3 increases in x_3. Thus, to find conditions for association of T_1, T_2 and T_3 we only need to find conditions such that

(4.9) a_2 increases in x_1,

(4.10) a_3 increases in x_2,

(4.11) a_3 increases in x_1.

As in Section 4.1 it can be shown that (4.9) is the same as $\psi_{2|1}(t_2|t_1) > \psi_{2|1}(t_2|t_1+u)$, $t_2 > t_1 + u > t_1 > 0$, or, more explicitly,

$$(4.12)\quad \Lambda_2(t_1) + \Lambda_2(t_2-t_1|T_1=t_1) > \Lambda_2(t_1+u) + \Lambda_2(t_2-t_1-u|T_1=t_1+u),\ t_2 > t_1+u > t_1 > 0,$$

or, by means of derivatives,

$\lambda_2(t_1) < \lambda_2(t_2|T_1=t_1,\cdot) - \frac{\partial}{\partial c_1}\Lambda_2(t_2-t_1|T_1=c_1)\big|_{c_1=t_1}$, $t_2 > t_1 > 0$. Similarly, (4.10) is the same as

$\psi_{3|1,2}(t_3|t_1,t_2) \geq \psi_{3|1,2}(t_3|t_1,t_2+u)$, $t_3 \geq t_2 + u \geq t_2 \geq t_1 \geq 0$, or more explicitly,

$$\text{(4.13)} \quad \Lambda_3(t_2-t_1|T_1=t_1) + \Lambda_3(t_3-t_2|T_1=t_1,\ T_2 = t_2) \geq$$
$$\Lambda_3(t_2+u-t_1|T_1=t_1) + \Lambda_3(t_3-t_2-u|T_1=t_1,\ T_2=t_2+u),\ t_3 \geq t_2 + u \geq t_2 \geq t_1 \geq 0,$$

or, by means of derivatives,

$\lambda_3(t_2|T_1=t_1,\cdot) \leq \lambda_3(t_3|T_1=t_1,T_2=t_2,\cdot) - \frac{\partial}{\partial c_2}\Lambda_3(t_3-t_2|T_1=t_1,T_2=c_2)\big|_{c_2=t_2}$,

$t_3 \geq t_2 \geq t_1 \geq 0$. Finally, (4.11) is the same as (here

$\tilde{a}_2(x_2,a) \equiv a + \Lambda_2^{-1}(x_2-\Lambda_2(a)|T_1=a), x_2 \geq 0,\ a \geq 0)$

$\psi_{3|1,2}(t_3|t_1,\tilde{a}_2(x_2,t_1)) \geq \psi_{3|1,2}(t_3|t_1+u,\tilde{a}_2(x_2,t_1+u))$,

$x_2 \geq 0,\ 0 \leq t_1 \leq t_1 + u \leq \tilde{a}_2(x_2,t_1) \leq \tilde{a}_2(x_2,t_1+u) \leq t_3$, or, more explicitly,

$$\Lambda_3(t_1) + \Lambda_3(\tilde{a}_2(x_2,t_1) - t_1|T_1 = t_1) + \Lambda_3(t_3-\tilde{a}_2(x_2,t_1)|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1))$$
$$\text{(4.14)} \quad \geq \Lambda_3(t_1+u) + \Lambda_3(\tilde{a}_2(x_2,t_1+u)-t_1-u|T_1=t_1+u)$$
$$+ \Lambda_3(t_3-\tilde{a}_2(x_2,t_1+u)|T_1=t_1+u,\ T_2=\tilde{a}_2(x_2,t_1+u))\ ,$$
$$x_2 \geq 0,\ 0 \leq t_1 \leq t_1 + u \leq \tilde{a}_2(x_2,t_1) \leq \tilde{a}_2(x_2,t_1+u) \leq t_3\ ,$$

(see Figure 4.2) or, by means of derivatives,

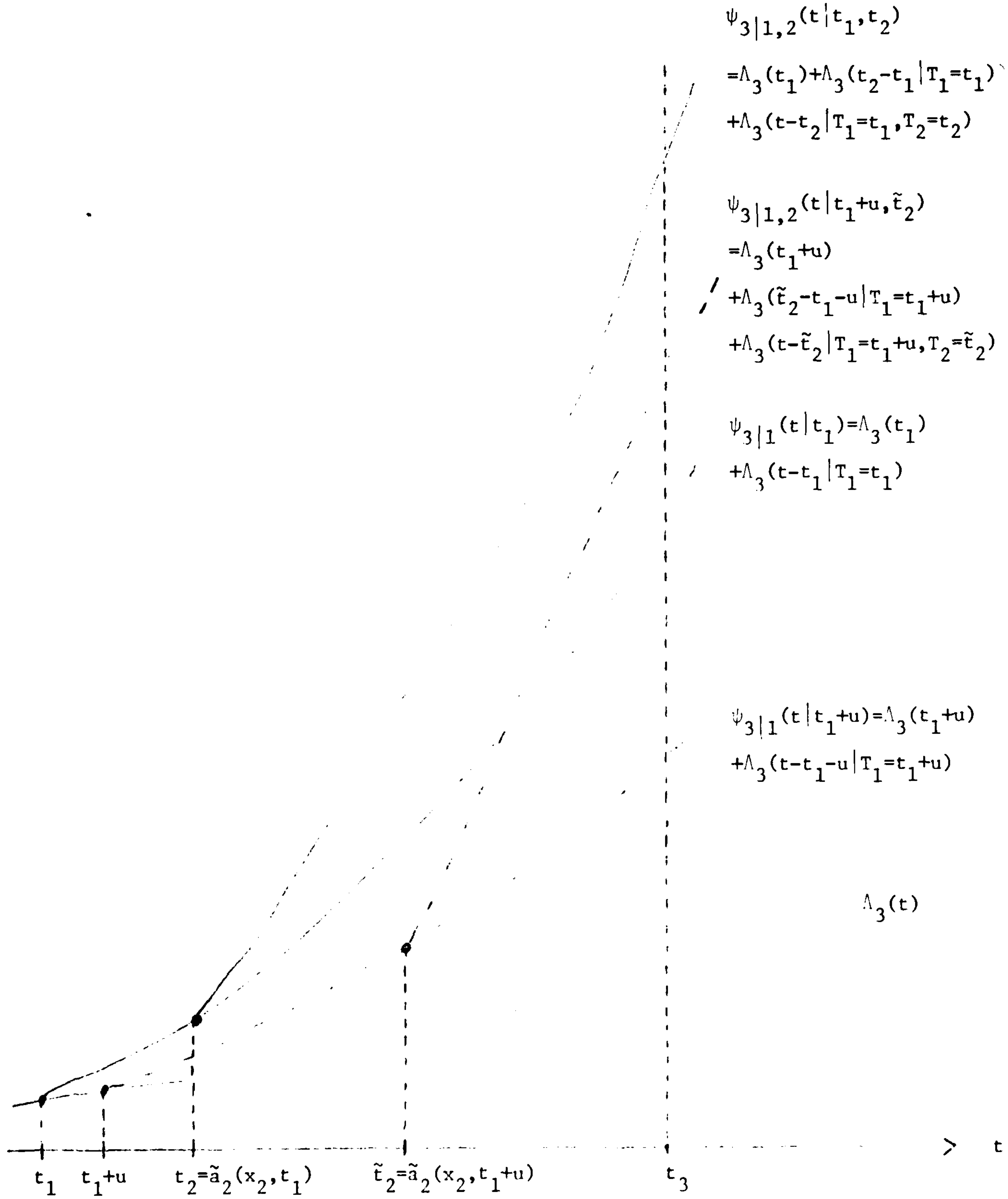

Figure 4.2. Illustration of (4.14).

$$(4.15)\qquad \lambda_3(t_1) + \lambda_3(t_2|T_1=t_1,\cdot)\,\frac{\partial\tilde{a}_2(x_2,a)}{\partial a}\Big|_{a=t_1} + \frac{\partial}{\partial c_1}\Lambda_3(t_2-t_1|T_1=c_1)\Big|_{c_1=t_1}$$

$$+ \frac{\partial}{\partial c_1}\Lambda_3(t_3-t_2|T_1=c_1,\ T=t_2)\Big|_{c_1=t_1}$$

$$+ \frac{\partial}{\partial c_2}\Lambda_3(t_3-t_2|T_1=t_1,\ T_2=c_2)\Big|_{c_2=t_2}\cdot\frac{\partial\tilde{a}_2(x_2,a)}{\partial a}\Big|_{a=t_1}$$

$$\leq \lambda_3(t_2|T_1=t_1,\cdot) + \lambda_3(t_3|T_1=t_1,T_2=t_2)\cdot\frac{\partial\tilde{a}_2(x_2,a)}{\partial a}\Big|_{a=t_1}$$

where in (4.15) $t_2 \equiv \tilde{a}_2(x_2,t_1)$.

In the statement of the next theorem we use the notation (see (3.4))

$$\tilde{a}_k(x_k,t_1,\ldots,t_{k-1}) \equiv t_{k-1} + \Lambda_k^{-1}[x_k - \psi_{k|1,\ldots,k-1}(t_{k-1}|t_1,\ldots,t_{k-1})|T_1=t_1,\ldots,T_{k-1}=t_{k-1}]$$

which describes, according to (3.4), how $\hat{T}_k$ is determined, given that $j_1 = 1,\ldots,j_{k-1}=k-1$, $j_k = k$ and that $X_k = x_k$, $\hat{T}_1 = t_1,\ldots,\hat{T}_{k-1} = t_{k-1}$.

Theorem 4.4. If $\psi_{i|1,2,\ldots,i-1}[t_i|t_1,t_2,\ldots,t_k,$ $\tilde{a}_{k+1}(x_{k+1},t_1,\ldots,t_k),\tilde{a}_{k+2}(x_{k+2},t_1,\ldots,t_k,\tilde{a}_{k+1}),\ldots,$ $\tilde{a}_{i-1}(x_{i-1},t_1,\ldots,t_k,\ \tilde{a}_{k+1},\ldots,\tilde{a}_{i-2})]$ (the arguments of some of the $\tilde{a}_m$'s are omitted) decreases in

$t_k \in \{t_k : t_k > t_{k-1},\ \tilde{a}_{k+1} > t_k,\ \tilde{a}_{\ell+1} > \tilde{a}_\ell,\ \ell = k+1,\ldots,i-1\}$ for all $1 \leq k < i \leq n$, $x_k \geq 0,\ldots,x_i \geq 0$, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_{k-1}$, and if the above condition holds for all permutations of the indices $1,2,\ldots,n$, then $T_1,\ldots,T_n$ are associated.

Sketch of the proof. The conceptualy simple but notationally involved proof of Theorem 4.4 is similar to the proof of Theorem 4.1 and we will omit the details. The idea of the proof is as follows: According to (3.4), given that $j_1 = 1,\ldots,j_{i-1} = i - 1$, $j_i = i$ and that $\hat{T}_1 = t_1,\ldots,\hat{T}_{i-1} = t_{i-1}$ and $X_i = x_i$, the value of $\hat{T}_i$ then is

$$t_i = t_{i-1} + \Lambda_i^{-1}[x_i - \psi_{i|1,\ldots,i-1}(t_{i-1}|t_1,\ldots,t_{i-1})|T_1 = t_1,\ldots,T_{i-1}=t_{i-1}] .$$

Equivalently, t_i is the solution of

$$(4.16) \qquad x_i = \psi_{i|1,\ldots,i-1}(t_i|t_1,\ldots,t_{i-1}) .$$

Association of $T_1,\ldots,T_n$ will follow if we show that t_i increases in $x_k (k < i)$. In order to do that, we can fix $t_1,\ldots,t_{k-1}$ and express t_k as a function of x_k and express $t_{k+1},\ldots,t_{i-1}$ as functions of t_k (and of $x_{k+1},\ldots,x_{i-1}$). Since, trivially, t_k increases in x_k, it suffices to show that t_i increases in t_k (recall that $t_1,\ldots,t_{k-1}$ and $x_k,\ldots,x_{i-1}$ are held fixed).

Rewrite (4.16) as

$$(4.17) \qquad x_i - \Lambda_i(t_1) - \sum_{\ell=2}^{k-1} \Lambda_i(t_\ell - t_{\ell-1}|T_1=t_1,\ldots,T_{\ell-1}=t_{\ell-1})$$

$$= \sum_{\ell=k}^{i-1} \Lambda_i(t_\ell - t_{\ell-1}|T_1=t_1,\ldots,T_{\ell-1}=t_{\ell-1}) + \Lambda_i(t_i - t_{i-1}|T_1=t_1,\ldots,T_{i-1}=t_{i-1})$$

and notice that t_i is determined as the solution of (4.17). Clearly

$$(4.18) \qquad \Lambda_i(u|T_1=t_1,\ldots,T_i=t_{i-1}) \text{ increases in } u > 0 .$$

The LHS (4.17) is fixed (i.e., it does not depend on t_k). If the condition of Theorem 4.4 holds, that is, if for a fixed t_i, RHS (4.17) decreases as t_k increases, then (using (4.18)) the solution t_i of (4.17) must increase as t_k increases. But this is what we wanted to prove. ||

Remark 4.5. Theorem 4.4 should be contrasted with Theorem 4.7, p. 146, of Barlow and Proschan (1975). They show that if for all t_i,

$$\text{(4.19)} \qquad P(T_i > t_i \mid T_1 = t_1,\ldots,T_{i-1} = t_{i-1}) \text{ increases in } t_1,\ldots,t_{i-1},\ i = 2,\ldots,n,$$

then $T_1,\ldots,T_n$ are associated. Their proof essentially constructs $T_1',\ldots,T_n'$ as in (3.10) and (3.11) and then argues that (4.19) implies that $T_1',\ldots,T_n'$ are increasing functions of $U_1,\ldots,U_n$ of (3.10) and (3.11). In proving our Theorem 4.4 we follow the same line of thought but apply it to $\hat{T}_1,\ldots,\hat{T}_n$ which arise from the total hazard construction described in (3.3) and (3.4).

Remark 4.6. Shaked and Shanthikumar (1984b) showed that if for disjoint sets $I, J \subset \{1,\ldots,n\}$ and fixed $\underline{t}_I$, $\hat{\underline{t}}_I$, $\underline{t}_J$ [such that $\underline{t}_I < \hat{\underline{t}}_I$] and $k \in \overline{I \cup J}$ (I or J may be empty),

$$\text{(4.20)} \qquad \begin{aligned} &\lambda_k\Big(\big(\bigvee_{i\in I} \hat{t}_i\big) \vee \big(\bigvee_{j\in J} t_j\big) + u \,\Big|\, \underline{T}_I = \underline{t}_I,\ \underline{T}_J = \underline{t}_J, \cdot\Big) \\ &> \lambda_k\Big(\big(\bigvee_{i\in I} \hat{t}_i\big) \vee \big(\bigvee_{j\in J} t_j\big) + u \,\Big|\, \underline{T}_I = \hat{\underline{t}}_I, \cdot\Big),\quad u > 0, \end{aligned}$$

then $T_1,\ldots,T_n$ are associated. Below it is argued that (4.20) implies the conditions of Theorem 4.4. Thus Theorem 4.4 provides a new route of proving Theorem 5.2 (and Remark 5.5) of Shaked and Shanthikumar (1984b).

To avoid messy notation we consider the case $n = 3$ and show that (4.20) implies (4.12), (4.13) and (4.14).

Proof that (4.20) ==> (4.12). From (4.20) we get

(4.21) $\lambda_2(t_1+v|\cdot) \leq \lambda_2(t_1+v|T_1=t_1,\cdot)$, $t_1 \geq 0$, $v \geq 0$.

Integrating (4.21) with respect to v over $[0,u]$ we obtain

(4.22) $\Lambda_2(t_1+u) - \Lambda_2(t_1) \leq \Lambda_2(u|T_1=t_1)$, $u \geq 0$, $t_1 \geq 0$.

From (4.20) we also get

(4.23) $\lambda_2(t_1+u+v|T_1=t_1+u,\cdot) \leq \lambda_2(t_1+u+v|T_1=t_1,\cdot)$, $t_1 \geq 0$, $u \geq 0$, $v \geq 0$.

Integrating (4.23) with respect to v over $[0,t_2-t_1-u]$ we obtain

(4.24) $\Lambda_2(t_2-t_1-u|T_1=t_1+u) \leq \Lambda_2(t_2-t_1|T_1=t_1) - \Lambda_2(u|T_1=t_1)$, $t_2 \geq t_1+u \geq t_1 \geq 0$.

Adding (4.22) and (4.24) and rearranging one obtains (4.12). ‖

Proof that (4.20) ==> (4.13). From (4.20) we get

(4.25) $\lambda_3(t_2+v|T_1=t_1,\cdot) \leq \lambda_3(t_2+v|T_1=t_1,T_2=t_2,\cdot)$, $t_1 \geq 0$, $t_2 \geq 0$, $u \geq 0$, $v \geq 0$.

Integrate (4.25) with respect to v over $[0,u]$ to obtain

(4.26) $\Lambda_3(t_2+u-t_1|T_1=t_1) - \Lambda_3(t_2-t_1|T_1=t_1) \leq \Lambda_3(u|T_1=t_1,T_2=t_2)$, $t_2+u \geq t_2 \geq t_1 \geq 0$.

Condition (4.20) also yields

(4.27) $\lambda_3(t_2+u+v|T_1=t_1,T_2=t_2+u,\cdot) \leq \lambda_3(t_2+u+v|T_1=t_1,T_2=t_2,\cdot)$, $t_1 \geq 0, t_2 \geq 0, u \geq 0, v \geq 0$.

Integrate (4.27) with respect to v over $[0,t_3-t_2-u]$ to obtain

(4.28) $$\Lambda_3(t_3-t_2-u|T_1=t_1,T_2=t_2+u) \leq \Lambda_3(t_3-t_2|T_1=t_1,T_2=t_2) - \Lambda_3(u|T_1=t_1,T_2=t_2),\quad t_3 \geq t_2+u \geq t_2 \geq t_1 \geq 0.$$

Add (4.26) and (4.28) and rearrange to obtain (4.13). ||

Proof that (4.20) ==> (4.14). The following follow from (4.20) [See Figure 4.2]:

(4.29) $$\lambda_3(t_1+v|\cdot) \leq \lambda_3(t_1+v|T_1=t_1,\cdot),\quad t_1 \geq 0,\ v \geq 0,$$

(4.30) $$\lambda_3(t_1+u+v|T_1=t_1+u,\cdot) \leq \lambda_3(t_1+u+v|T_1=t_1,\cdot),\quad t_1 \geq 0,\ u \geq 0,\ v \geq 0,$$

(4.31) $$\lambda_3(\tilde{a}_2(x_2,t_1)+v|T_1=t_1+u,\cdot) \leq \lambda_3(\tilde{a}_2(x_2,t_1)+v|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1),\cdot),\quad t_1 \geq 0,\ u \geq 0,\ v \geq 0,\ x_2 \geq 0,$$

(4.32) $$\lambda_3(\tilde{a}_2(x_2,t_1+u)+v|T_1=t_1+u,\ T_2=\tilde{a}_2(x_2,t_1+u),\cdot) \leq \lambda_3(\tilde{a}_2(x_2,t_1+u)+v|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1),\cdot),\quad t_1 \geq 0,\ x_2 \geq 0,\ u \geq 0,\ v \geq 0.$$

Integrate (4.29) with respect to v over $[0,u]$ to obtain

(4.33) $$\Lambda_3(t_1+u) - \Lambda_3(t_1) \leq \Lambda_3(u|T_1=t_1),\quad 0 \leq t_1 \leq t_1+u.$$

Integrate (4.30) with respect to v over $[0,\ \tilde{a}_2(x_2,t_1) - t_1 - u]$ to obtain

(4.34) $$\Lambda_3(\tilde{a}_2(x_2,t_1)-t_1-u|T_1=t_1+u) \leq \Lambda_3(\tilde{a}_2(x_2,t_1)-t_1|T_1=t_1) - \Lambda_3(u|T_1=t_1),\quad 0 \leq t_1 \leq t_1+u \leq \tilde{a}_2(x_2,t_1).$$

Integrate (4.31) with respect to v over $[0,\ \tilde{a}_2(x_2,t_1+u)-\tilde{a}_2(x_2,t_1)]$ to obtain

$$(4.35)\quad \Lambda_3(\tilde{a}_2(x_2,t_1+u)-t_1-u|T_1=t_1+u) - \Lambda_3(\tilde{a}_2(x_2,t_1)-t_1-u|T_1=t_1+u)$$
$$\leq \Lambda_3(\tilde{a}_2(x_2,t_1+u)-\tilde{a}_2(x_2,t_1)|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1)),$$
$$0 \leq t_1 \leq t_1 + u \leq \tilde{a}_2(x_2,t_1) \leq \tilde{a}_2(x_2,t_1+u).$$

Finally integrate (4.32) with respect to v over $[0, t_3 - \tilde{a}_2(x_2,t_1+u)]$ to obtain

$$(4.36)\quad \Lambda_3(t_3-\tilde{a}_2(x_2,t_1+u)|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1+u))$$
$$\leq \Lambda_3(t_3-\tilde{a}_2(x_2,t_1)|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1))$$
$$- \Lambda_3(\tilde{a}_2(x_2,t_1+u)-\tilde{a}_2(x_2,t_1)|T_1=t_1,T_2=\tilde{a}_2(x_2,t_1)),$$
$$0 \leq t_1 \leq t_1+u \leq \tilde{a}_2(x_2,t_1) \leq \tilde{a}_2(x_2,t_1+u) \leq t_3.$$

Add (4.33) - (4.36) and rearrange to obtain (4.14). ||

5. An application: stochastic ordering.

5.1. The bivariate case.

Let (S_1,S_2) and (T_1,T_2) be two nonnegative absolutely continuous random vectors. The corresponding hazard rates and cumulative hazard functions will be denoted as follows:

$$q_i(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < S_i \leq t + \Delta t | S_1 > t,\ S_2 > t\},\ t \geq 0,\ i = 1,2,$$
$$r_i(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_i \leq t + \Delta t | T_1 > t,\ T_2 > t\},\ t \geq 0,\ i = 1,2,$$
$$Q_i(t) = \int_0^t q_i(u)du,\ t \geq 0,\ i = 1,2,$$
$$R_i(t) = \int_0^t r_i(u)du,\ t \geq 0,\ i = 1,2,$$
$$q_1(t|t_2) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < S_1 \leq t + \Delta t | S_1 > t,\ S_2 = t_2\},\ t \geq t_2 \geq 0,$$

$$q_2(t|t_1) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < S_2 < t + \Delta t | S_1 = t_1, S_2 > t\},\ t > t_1 > 0,$$
$$r_1(t|t_2) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_1 < t + \Delta t | T_1 > t, T_2 = t_2\},\ t > t_2 > 0,$$
$$r_2(t|t_1) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_2 < t + \Delta t | T_1 = t_1, T_2 > t\},\ t > t_1 > 0,$$

$$Q_1(t|t_2) = \int_{t_2}^{t_2+t} q_1(u|t_2)du,\ t > 0,\ t_2 > 0,$$

$$Q_2(t|t_1) = \int_{t_1}^{t_1+t} q_2(u|t_1)du,\ t > 0,\ t_1 > 0,$$

$$R_1(t|t_2) = \int_{t_2}^{t_2+t} r_1(u|t_2)du,\ t > 0,\ t_2 > 0,$$

$$R_2(t|t_1) = \int_{t_1}^{t_1+t} r_2(u|t_1)du,\ t > 0,\ t_1 > 0.$$

The inverses of $Q_i(\cdot)$, $Q_i(\cdot|t)$, $R_i(\cdot)$ and $R_i(\cdot|t)$ are defined in an obvious manner as in Section 2.

Let X_1 and X_2 be independent standard exponential random variables. Define $\hat{S}_1$, $\hat{S}_2$ as follows: On $\{Q_1^{-1}(X_1) < Q_2^{-1}(X_2)\}$ let

(5.1) $\quad \hat{S}_1 \equiv Q_1^{-1}(X_1),$

(5.2) $\quad \hat{S}_2 \equiv Q_1^{-1}(X_1) + Q_2^{-1}(X_2 - Q_2(Q_1^{-1}(X_1))|Q_1^{-1}(X_1)),$

and on $\{Q_1^{-1}(X_1) > Q_2^{-1}(X_2)\}$ let

(5.3) $\quad \hat{S}_1 \equiv Q_2^{-1}(X_2) + Q_1^{-1}(X_1 - Q_1(Q_2^{-1}(X_2))|Q_2^{-1}(X_2)),$

(5.4) $\quad \hat{S}_2 \equiv Q_2^{-1}(X_2).$

Similarly on $\{R_1^{-1}(X_1) < R_2^{-1}(X_2)\}$ let

$$(5.5)\qquad \hat{T}_1 \equiv R_1^{-1}(X_1),$$
$$(5.6)\qquad \hat{T}_2 \equiv R_1^{-1}(X_1) + R_2^{-1}(X_2 - R_2(R_1^{-1}(X_1))|R_1^{-1}(X_1)),$$

and on $\{R_1^{-1}(X_1) > R_2^{-1}(X_2)\}$ let

$$(5.7)\qquad \hat{T}_1 \equiv R_2^{-1}(X_2) + R_1^{-1}(X_1 - R_1(R_2^{-1}(X_2))|R_2^{-1}(X_2)),$$
$$(5.8)\qquad \hat{T}_2 \equiv R_2^{-1}(X_2).$$

<u>Theorem 5.1.</u> Let $(\hat{S}_1,\hat{S}_2)$ and $(\hat{T}_1,\hat{T}_2)$ be defined as in (5.1) - (5.8) where X_1 and X_2 are independent exponential random varibles. Then

$$(\hat{S}_1,\hat{S}_2) \overset{st}{=} (S_1,S_2),$$
$$(\hat{T}_1,\hat{T}_2) \overset{st}{=} (T_1,T_2).$$

<u>Proof:</u> Apply Theorem 2.1 twice. ‖

In (5.1) - (5.8) we use the total hazard construction twice: to construct $\hat{\underline{S}}$ and to construct $\hat{\underline{T}}$. Note that we use the same X_1 and X_2 for both constructions. Thus, roughly speaking, we put $\underline{S}$ and $\underline{T}$ on the same probability space. This enables us to compare them realization-wise as is done, e.g., in the next theorem.

<u>Theorem 5.2.</u> If

$$(5.9)\qquad Q_1(w) \geq R_1(w),\ w \geq 0,$$
$$(5.10)\qquad Q_2(w) \geq R_2(w),\ w \geq 0,$$
$$(5.11)\qquad Q_2(s_1) + Q_2\ (w-s_1|s_1) \geq R_2(w),\ w \geq s_1 \geq 0,$$
$$(5.12)\qquad Q_1(s_2) + Q_1\ (w-s_2|s_2) \geq R_1(w),\ w \geq s_2 \geq 0,$$

$$(5.13)\quad Q_2(s_1) + Q_2(w-s_1|s_1) \geq R_2(t_1) + R_2(w-t_1|t_1),\ w \geq t_1 \geq s_1 \geq 0,$$

$$(5.14)\quad Q_1(s_2) + Q_1(w-s_2|s_2) \geq R_1(t_2) + R_1(w-t_2|t_2),\ w \geq t_2 \geq s_2 \geq 0,$$

then

$$(5.15)\quad (S_1,S_2) \overset{st}{\leq} (T_1,T_2).$$

Remark 5.3. The conditions of Theorem 5.2 simply state that at any time w (no matter what the previous history is) the cumulative hazard of S_i is larger then the cumulative hazard of T_i, $i = 1,2$. The proof of Theorem 5.2 below uses the fact that, since the total cumulative hazards of $\hat{S}_i$ and $\hat{T}_i$ by the time they failed must be equal (to X_i), then necessarily $\hat{S}_i \leq \hat{T}_i$ a.s., $i = 1,2$ (see Figure 5.1 for a typical realization when $Q_1^{-1}(X_1) \leq Q_2^{-1}(X_2)$).

Proof of Theorem 5.2. We will show that

$$(5.16)\quad \hat{S}_1 \leq \hat{T}_1 \qquad \text{a.s.},$$

$$(5.17)\quad \hat{S}_2 \leq \hat{T}_2 \qquad \text{a.s.},$$

and the result then follows from Theorem 5.1 and (1.5).

Let X_1 and X_2 be independent standard exponential random variables. First consider the case $Q_1^{-1}(X_1) \leq Q_2^{-1}(X_2)$. Then

$$\hat{S}_1 = Q_1^{-1}(X_1),$$

$$(5.18)\quad \hat{S}_2 = \hat{S}_1 + Q_2^{-1}(X_2 - Q_2(\hat{S}_1)|\hat{S}_1).$$

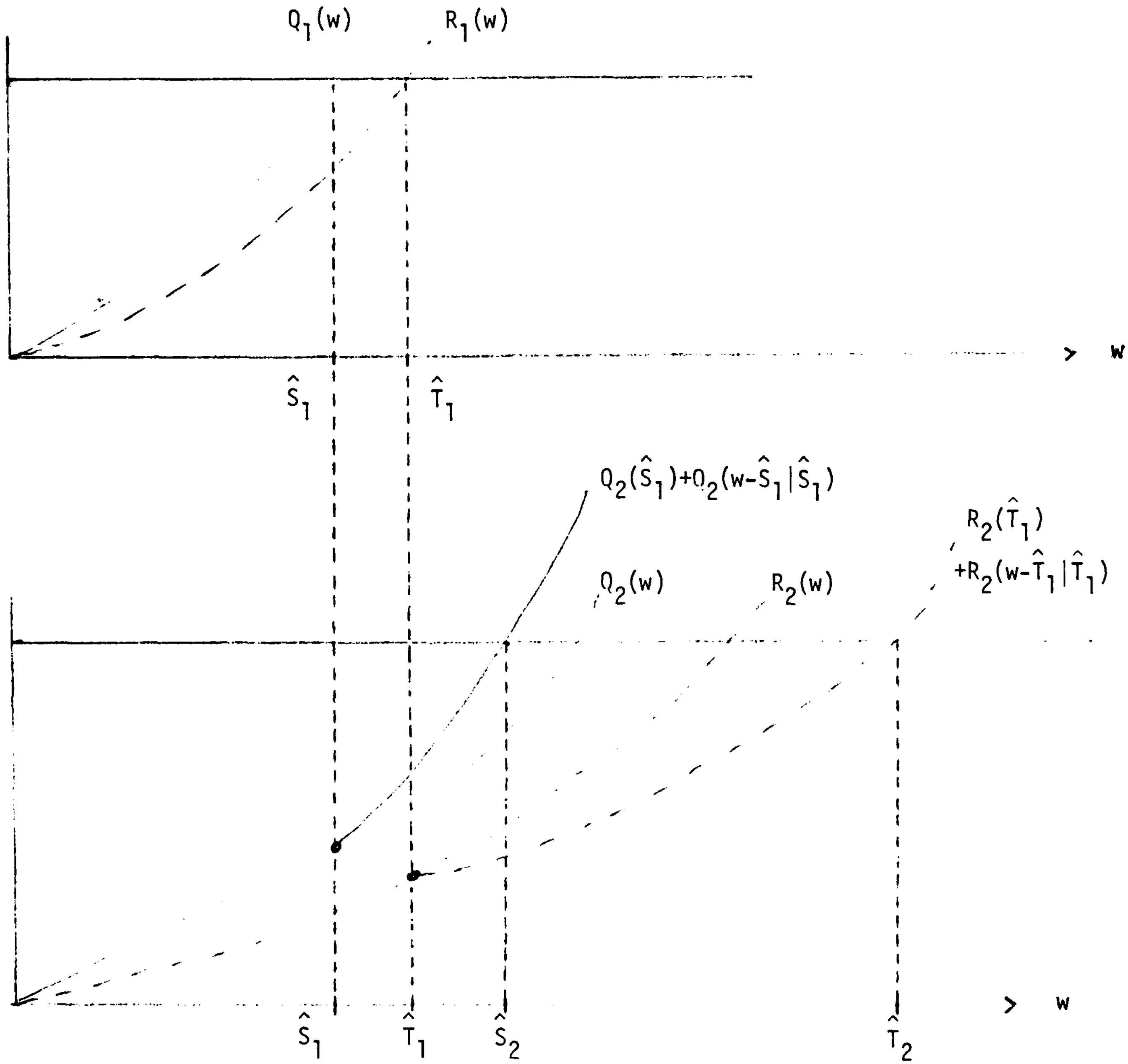

Figure 5.1. Illustration of (5.9), (5.11) and (5.13).

If $R_1^{-1}(X_1) < R_2^{-1}(X_2)$ then $\hat{T}_1 = R_1^{-1}(X_1) \underset{(5.9)}{>} Q_1^{-1}(X_1) = \hat{S}_1$ and (5.16) follows. Also, in this case

$$(5.19) \qquad \hat{T}_2 = \hat{T}_1 + R_2^{-1}(X_2 - R_2(\hat{T}_1) | \hat{T}_1) .$$

Thus $\hat{T}_2 \underset{(5.19)}{=} \inf\{w > \hat{T}_1 : R_2(\hat{T}_1) + R_2(w - \hat{T}_1 | \hat{T}_1) > X_2\}$

$\underset{(5.13)}{>} \inf\{w > \hat{S}_1 : Q_2(\hat{S}_1) + Q_2(w - \hat{S}_1 | \hat{S}_1) > X_2\} \underset{(5.18)}{=} \hat{S}_2$ and (5.17) follows.

If $R_1^{-1}(X_1) > R_2^{-1}(X_2)$ then

$$(5.20) \qquad \hat{T}_2 = R_2^{-1}(X_2) ,$$

$$(5.21) \qquad \hat{T}_1 = \hat{T}_2 + R_1^{-1}(X_1 - R_1(\hat{T}_2) | \hat{T}_2) .$$

By assumption, $\hat{S}_1 < \hat{S}_2$. Thus,

$$(5.22) \qquad \hat{S}_1 < \hat{S}_2 \underset{(5.18)}{=} \inf\{w > \hat{S}_1 : Q_2(\hat{S}_1) + Q_2(w - \hat{S}_1 | \hat{S}_1) > X_2\}$$
$$\underset{(5.11)}{<} \inf\{w > \hat{S}_1 : R_2(w) > X_2\} .$$

Also, $R_2(\hat{S}_1) \underset{(5.10)}{<} Q_2(\hat{S}_1) = Q_2(Q_1^{-1}(X_1)) < X_2$, where the second inequality follows from the assumption $Q_1^{-1}(X_1) < Q_2^{-1}(X_2)$. Thus, since $R_2(w)$ increases in $w > 0$,

$$(5.23) \qquad \inf\{w > \hat{S}_1 : R_2(w) > X_2\} = \inf\{w > 0 : R_2(w) > X_2\}$$
$$\underset{(5.20)}{=} \hat{T}_2 \underset{(5.21)}{<} \hat{T}_1 .$$

Combining (5.22) and (5.23) it follows that $\hat{S}_1 < \hat{S}_2 < \hat{T}_2 < \hat{T}_1$ and both (5.16) and (5.17) follow.

The proof of (5.16) and (5.17) for the case $Q_1^{-1}(X_1) > Q_2^{-1}(X_2)$ is similar, using (5.10), (5.19), (5.12) and (5.9) instead of (5.9), (5.13), (5.11) and (5.10) respectively. ||

5.2. The multivariate case.

Let $\underline{S} = (S_1,\ldots,S_n)$ and $\underline{T} = (T_1,\ldots,T_n)$ be nonnegative absolutely continuous random vectors. Denote by $q_i(s|\underline{S}_J = \underline{s}_J,\cdot)$ the conditional hazard rates of $\underline{S}$ defined as in (3.1) and denote by $Q_i(s|\underline{S}_J = \underline{s}_J)$ the conditional cumulative hazards of $\underline{S}$ defined as in (3.2). Similarly denote the conditional hazard rates and cumulative hazards of $\underline{T}$ by $r_i(t|\underline{T}_J = \underline{t}_J,\cdot)$ and $R_i(t|\underline{T}_J = \underline{t}_J)$.

Using the total hazard construction (see Section 3) one can express $\hat{\underline{S}}$ and $\hat{\underline{T}}$ (such that $\hat{\underline{S}} \overset{st}{=} \underline{S}$ and $\hat{\underline{T}} \overset{st}{=} \underline{T}$) as functions of the same independent standard exponential random variables $X_1,\ldots,X_n$. Using these $\hat{\underline{S}}$ and $\hat{\underline{T}}$ one can prove the following result using the method of the proof of Theorem 5.2 (but with more involved notation). We omit the details.

Theorem 5.4. If for $1 \le \ell \le j \le n$, $0 \le s_1 \le \ldots \le s_j$, $0 \le t_1 \le \ldots \le t_\ell$, $0 \le s_i \le t_i$, $i = 1,\ldots,\ell$, and all permutations $\underline{\pi}$ of $(1,\ldots,n)$,

$$
\begin{aligned}
(5.24) \qquad & \sum_{i=1}^{j-1} Q_k(s_{i+1}-s_i|S_{\pi(1)} = s_1,\ldots,S_{\pi(i)} = s_i) \\
& \quad + Q_k(w-s_j|S_{\pi(1)} = s_1,\ldots,S_{\pi(j)} = s_j) \\
& \ge \sum_{i=1}^{\ell-1} R_k(t_{i+1}-t_i|T_{\pi(1)} = t_1,\ldots,T_{\pi(i)} = t_i) \\
& \quad + R_k(w-t_\ell|T_{\pi(1)} = t_1,\ldots,T_{\pi(\ell)} = t_\ell)
\end{aligned}
$$

whenever $w > s_j \vee t_\ell$ (where empty sums are identically zero) then

$$(5.25) \qquad \underline{S} \overset{st}{\le} \underline{T}.$$

The proof of Theorem 5.4 consists of constructing $\hat{\underline{S}}$ and $\hat{\underline{T}}$ by (3.3) and (3.4) using the same $X_1,\ldots,X_n$ (that is, putting $\underline{S}$ and $\underline{T}$ on the same probability space) and noticing that (5.24) implies that realization-wise $\hat{\underline{S}} \leq \hat{\underline{T}}$. The result then follows from (1.5).

Remark 5.5. Using the standard construction (3.10) and (3.11) one can show the following analog of Theorem 5.4 (see, e.g. Veinott (1965) or Arjas and Lehtonen (1978)): If

(5.26) $$S_1 \leq^{st} T_1$$

and for $s_1 \leq t_1,\ldots,s_{i-1} \leq t_{i-1}$,

(5.27) $$[S_i | S_1 = s_1,\ldots,S_{i-1} = s_{i-1}] \leq^{st} [T_i | T_1 = t_1,\ldots,T_{i-1} = t_{i-1}],$$

$i = 2,3,\ldots,n$, then $\underline{S} \leq^{st} \underline{T}$. The idea of the proof of (5.26) + (5.27) ==> (5.25) is the same as the proof of Theorem 5.4: Using the same $U_1,\ldots,U_n$ of (3.10) and (3.11), put $\underline{S}$ and $\underline{T}$ on the same probability space and note that (5.26) and (5.27) imply that realization-wise $\underline{S}' \leq \underline{T}'$.

Remark 5.6. Shaked and Shanthikumar (1984b) proved that if for all disjoint sets $I, J \subset \{1,\ldots,n\}$ such that $I \cup J \neq \emptyset$ and for all fixed $\underline{v}_J \geq 0\underline{e}$ the following holds:

(5.28) $$\begin{aligned} &q_k(u | \underline{S}_I = \underline{v}_I, \underline{S}_J = \underline{v}_J, \cdot), \\ &\geq r_k(u | \underline{T}_I = \hat{\underline{v}}_I, \cdot), \quad u \geq (\bigvee_{j\in J} v_j) \vee (\bigvee_{i\in I} \hat{v}_i), \end{aligned}$$

whenever $\underline{\hat{v}}_I > \underline{v}_I$, $u > 0$ and $k \in \overline{I \cup J}$ (I or J may be empty) then $\underline{S} \overset{st}{\leq} \underline{T}$.

Theorem 5.4 provides a new way of proving their result. In fact Theorem 5.4 is a stronger result than Theorem 3.1 and 3.4 of Shaked and Shanthikumar (1984b) because, as will be argued shortly, (5.28) implies (5.24).

To see that (5.28) implies (5.24) suppose that in (5.24), $\underline{\pi} = (1,\ldots,n)$ and order the j s_i's and ℓ t_i's of (5.24) in an increasing order $0 \leq v_1 \leq \ldots \leq v_{j+\ell}$ where each v_m is either s_i or t_i for some i. Also set $v_0 \equiv 0$ and $v_{j+\ell+1} \equiv w$ (the w of (5.24)). Consider some interval $[v_m, v_{m+1})$. Let $I = \{i: t_i \leq v_m\}$, $J = \{i \in \overline{I},\ s_i \leq v_m\}$. Note that if $i \in I$ then $s_i \leq v_m$ because in (5.24), $s_i \leq t_i$ for $i = 1,\ldots,\ell$. Let $u \in [v_m, v_{m+1})$ and apply (5.28) with the above I and J and $\underline{v}_I = \underline{s}_I$, $\underline{\hat{v}}_I = \underline{t}_I$, $\underline{v}_J = \underline{s}_J$ to obtain

$$(5.29) \qquad q_k(u | \underline{S}_I = \underline{s}_I, \underline{S}_J = \underline{s}_J, \cdot) \geq r_k(u | \underline{T}_I = \underline{t}_I, \cdot) .$$

Integrating (5.29) with respect to u over $[v_m, v_{m+1})$ and adding the resulting integral inequalities over $m = 0,\ldots,j+\ell$ one obtains (5.24) with $\underline{\pi} = (1,2,\ldots,n)$. The proof for other permutations $\underline{\pi}$ is similar.

For example, to obtain (5.9) note that (5.28) implies

$$(5.30) \qquad q_1(u | \cdot) \leq r_1(u | \cdot),\ u \geq 0.$$

Integrate (5.30) with respect to u over $[0,w)$ to obtain (5.9). To obtain (5.11) note that (5.28) implies

$$(5.31) \qquad q_1(u | S_1 = s_1, \cdot) \leq r_1(u | \cdot),\ u \geq s_1 .$$

Integrate (5.30) with respect to u over $[0,s_1)$ and integrate (5.31) with respect to u over $[s_1,w)$ and add the two resulting integral inequalities to obtain (5.11).

6. Further applications.

6.1. Multivariate increasing failure rate average (MIFRA) distributuions.

Theorems 4.1, 4.3 and 4.4 give conditions under which the distribution of $T_1,\ldots,T_n$ can be expressed as the distribution of increasing functions of independent exponential random variables. If these increasing functions are also subhomogeneous (a function $g: R_+^n \to R_+$ is subhomogeneous if $g(\alpha \underline{t}) < \alpha\, g(\underline{t})$ for all $\alpha \in [0,1]$, $\underline{t} > 0$, see, e.g., Marshall and Shaked (1982)) then $\underline{T}$ satisfies the MIFRA condition of Block and Savits (1980). For example in (2.21) [when $\alpha < \alpha'$, $\beta < \beta'$] $\hat{T}_1$ and $\hat{T}_2$ are expressed as incresing subhomogeneous functions of X_1 and X_2. Hence (T_1,T_2) of Example 1 (when $\alpha < \alpha'$, $\beta < \beta'$) is MIFRA. This result has been obtained also by Marshall and Shaked (1982) and Shaked (1984). A special case of this result can be found in Block and Savits (1980).

6.2. Variance reduction in simulation of dependent variables.

Let $\underline{S} = (S_1,\ldots,S_n)$ and $\underline{T} = (T_1,\ldots,T_n)$ be random vectors and let $g: R^n \to R$ and $h: R^n \to R$ be monotone in the same (or the opposite) direction. Due to theoretical or technical reasons, the expected value

$$(6.1) \qquad E[g(\underline{S}) - h(\underline{T})]$$

may be hard to compute in some applications. One possible recourse is a simulation of $g(\underline{S})$ and $h(\underline{T})$. That is, k independent replications of

$\underline{S}$ and $\underline{T}$ are generated using pseudo-random numbers, and (6.1) is then estimated by averaging the k realizations of $g(\underline{S}) - h(\underline{T})$.

Rubinstein, Samorodnitski and Shaked (1985) have considered an efficient method of simulating $g(\underline{S})$ and $h(\underline{T})$ when the distributions of $\underline{S}$ and $\underline{T}$ satisfy (4.19). Their method is based on the fact that when (4.19) holds, then $g(\underline{S})$ and $h(\underline{T})$ can be represented as increasing functions of independent uniform [0,1] random variables, using the standard construction (3.10) and (3.11). Then, by putting $\underline{S}$ and $\underline{T}$ on the same probability space, one can reduce the variance of the Monte Carlo estimate of $E(g(\underline{S})-h(\underline{T}))$.

The same idea may apply for random vectors with distributions satisfying the condition of Theorem 4.4. Under this condition too it is possible to represent $g(\underline{S})$ and $h(\underline{T})$ as increasing functions of independent random variables, put them on the same probability space and reduce the variance of the Monte Carlo estimate.

In some applications, even if both (4.19) and the condition of Theorem 4.4 hold, the total hazard construction (3.3) and (3.4) may yield simpler expressions than the standard construction (3.10) and (3.11). In such cases use of the total hazard construction is prefarable. A study of these ramifications of the total hazard construction is planned.

6.3. Multi-unit imperfect repair.

Shaked and Shanthikumar (1984a,b) considered a model for imperfect repair of multi-unit systems. In that model, n units (whose original lives $T_1,\ldots,T_n$ have absolutely continuous distribution) start to live at the same time. Upon failure an item undergoes a repair and is scrapped if the repair is unsuccessful. If i items $(i = 0,1,\ldots,n-1)$ have already been scrapped, then, with probability p_{i+1} the repair is successful and the item continues to function - but it is only as good as it was just before the repair - and the other items "do not know" about these failure and repair. With probability $1-p_{i+1}$ the repair is unsuccessful and the item is scrapped.

Mathematically, if the original lives have the conditional hazard rates $\lambda_k(\cdot|\underline{T}_I = \underline{t}_I,\cdot)$, $k \in \bar{I}$, then the resulting lives $\tilde{T}_1,\ldots,\tilde{T}_n$ have the conditional hazard rates $\tilde{\lambda}_k$ given by

$$(6.2) \qquad \tilde{\lambda}_k(\cdot|\underline{\tilde{T}}_I = \underline{t}_I,\cdot) = p_{|I|+1}\lambda_k(\cdot|\underline{T}_I = \underline{t}_I,\cdot), \; k \in \bar{I},$$

where $|I|$ is the cardinality of I.

From (6.2) it follows (see (3.2)) that $\tilde{\Lambda}_k(t|\underline{\tilde{T}}_I = \underline{t}_I)$, $k \in \bar{I}$ -- the hazard accumulated by T_k during the time interval $(\bigvee_{i\in I} t_i, \bigvee_{i\in I} t_i + t]$, $t > 0$, -- is given by

$$(6.3) \qquad \tilde{\Lambda}_k(t|\underline{\tilde{T}}_I = \underline{t}_I) = p_{|I|+1}\,\Lambda_k(t|\underline{T}_I = \underline{t}_I), \; k \in \bar{I}.$$

Using (6.3), various results of this paper can be restated for $\tilde{T}_1,\ldots,\tilde{T}_n$. For example, if $p_1 = p_2 = \ldots = p_n$ and the Λ_k's satisfy the conditions of Theorem 4.4 then $\tilde{T}_1,\ldots,\tilde{T}_n$ are associated. Similarly, two random vectors resulting from application of imperfect repair can be compared stochastically if the original random vectors satisfy the conditions of Theorem 5.4. Proposition 6.6 of Shaked and Shanthikumar (1984b) can be proven using these ideas.

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After the paper was written, when it was shuttling between Arizona and Berkeley for final touch-ups, we learned about the papers by Norros (1983, 1984) which contain results similar to some of the results of the present paper. For example, the main mathematical tool in Norros (1984) is a "compensator representation" which is essentially the same as our "total hazard construction". Results which are similar to Theorems 4.4 of 5.4 of the present paper, as well as conditions for MIFRA (see Section 6.1 of the present paper) and a notion of α-improvement which is related to our definition of imperfect repair, can be found in Norros (1984). Also, we learned about Aalen and Hoem (1978) and Jacobsen (1982) from Norros (1984). We thank Ilkka Norros for providing us with Norros (1983, 1984) and we thank Elja Arjas who initiated our correspondence with Norros.

References.

[1] Aalen, O. O. and Hoem, J. M. (1978). Random time changes for multivariate counting processes. Scand. Actuarial J., 1978, 81-101.

[2] Arjas, E. (1981). The failure and hazard processes in multivariate reliability systems. Math. Oper. Res., 6, 551-562.

[3] Arjas, E. and Lehtonen, T. (1978). Approximating many server queues by means of single server queues. Math. Oper. Res., 3, 205-223.

[4] Arjas, E. and Norros, I. (1984). Life lengths and association: a dynamic approach. Math. Oper. Res., 9, 151-158.

[5] Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

[6] Block, H. W. and Savits, T. H. (1980). Multivariate IFRA distributions. Ann. Prob., 8, 793-801.

[7] Cox, D. R. (1972). Regression models and life tables (with discussion). J. R. Statist. Soc., B, 34, 187-202.

[8] Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables, with applications. Ann. Math. Statist., 38, 1966-1974.

[9] Freund, J. E. (1961). A bivariate extension of the exponential distribution. J. Amer. Statist. Assoc., 56, 971-977.

[10] Jacobsen, M. (1982). Statistical Analysis of Counting Processes, Lecture Notes in Statistics, 12, Springer Verlag, New York.

[11] Johnson, N. L. and Kotz, S. (1972). Distributions in Statistics, Continuous Multivariate Distributions. Wiley, New York.

[12] Law, A. M. and Kelton, D. W. (1982). Simulation Modeling and Analysis. McGraw Hill, New York.

[13] Malik, H. J. and Abraham, B. (1973). Multivariate logistic distributions. Ann. Statist., 1, 588-590.

[14] Marshall, A. W. and Shaked, M. (1982). A class of multivariate new better than used distributions. Ann. Prob., 10, 259-264.

[15] Norros, I. (1983). Systems weakened by failures. Submitted for publication.

[16] Norros, I. (1984). A compensator representation of multivariate life length distributions, with applications. Submitted for publication.

[17] Rubinstein, R. Y. (1981). Simulation and the Monte Carlo Method. Wiley, New York.

[18] Rubinstein, R. Y., Samorodnitsky, G. and Shaked, M. (1985). Antithetic variates, multivariate dependence and simulation of stochastic systems. Management science, to appear.

[19] Schechner, Z. (1984). A load-sharing model: the linear breakdown rule. Naval Res. Log. Quart., 31, 137-144.

[20] Shaked, M. (1977). A concept of positive dependence for exchangeable random variables. Ann. Statist., 5, 505-515.

[21] Shaked, M. (1984). Extensions of the Freund distribution with applications in reliability theory. Oper. Res., 32, 917-925.

[22] Shaked, M. and Shanthikumar, J. G. (1984a). Multivariate imperfect repair. Technical Report, Department of Mathematics, University of Arizona.

[23] Shaked, M. and Shanthikumar, J. G. (1984b). Multivariate hazard rates and stochastic ordering. Technical Report, Department of Mathematics, University of Arizona.

[24] Veinott, R. (1965). Optimal policy in a dynamic, single product, nonstationary, inventory model with several demand classes. Oper. Res., 13, 761-778.

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